Pump ED 101

Why Newton Invented Calculus - (A Little Pump Math)

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Introduction

Have you ever noticed that some four letter words have more impact that others? We consider many of them to be obscene but there is one - the "M" word - that goes beyond obscenity and can even instill fear. The "M" word is Math and it is still illegal, in some states, to yell this word in a crowded theater. Why is this? In elementary school addition and subtraction came pretty easily and was often considered to be fun. Even multiplication and division seldom gave us headaches. But as we entered high school and progressed to algebra, geometry, and maybe even trigonometry eyes began to glaze over and we, somehow, learned something that was unintended - - avoid math whenever possible!

I believe that the real reason so many of us did our best to avoid higher level math, both then and today, is a direct result of the way it is taught. In elementary school we were taught the right way - - by example.

If Johnny has three apples but gives two away, how many does he have left?

Had we learned this in high school it would have been presented, rather blandly, as 3 – 2 = 1. But in grade school, not only did we learn the answer but also not to give away more than two apples if you want one for yourself! Practical examples are the only way we can truly see the positive aspects of math.

Now, I attended a very small high school back in the late fifties and early sixties so I probably got a little more personal attention than some of the big school graduates. But still, I remember very few, if any, concrete examples of why algebra and geometry might be useful at some point in my life. I could only assume that my chances for success would be greater if I learned to compute the area of a circle. And unfortunately, this method of mathematical presentation did not end in high school. I completed my first semester of college calculus without a single clue as to why I was there! In retrospect, the best math instruction I ever encountered was in my physics and physical chemistry classes. There, advanced math was presented in much the same way simple math was taught in elementary school - - by the use of practical examples.

In addition to poor presentation techniques, yet another hurdle surfaced. If you had the misfortune of attending middle school or high school during the mid sixties and early seventies you may have been a victim of the "new math". New math was the result of a brain stem storming session by the National Science Foundation and was, by all accounts, a dismal failure. Its basis was that of set theory and something known as the commutative property. Its primary focus was on math as a concept, and getting the right answer was purely secondary! Recently, I visited a web site that compared new math to the traditional teaching techniques that we should have seen in middle school. Consider the following.

Traditional Math – A logger sells a truck load of logs for $100. His cost to produce the logs was $70. How much profit did he make?

a) Profit = Sales Price – Cost of Production

New Math – A logger exchanges set L (logs) for set M (money). The cardinality of set M is 100 and each element is worth $1.

a) Make 100 dots representing the elements in set M
b) Set C (cost) has 30 fewer points than set M. Make C a subset of M
c) What is the cardinality of the new set P?

Believe it or not, this was the teaching trend during this period. [1] Now in the nineties things changed again but, this time it was all about self esteem. Often called the "no dummy left behind act", it placed the child's feelings ahead of his or her learning. An example of this technique follows.

Dumbed Down Math – A logger sells a truck load of logs for $100. His cost to produce the logs was $70 and his profit was $30.

a) Underline the number 30

And, we would not be complete without illustrating the techniques of the environmentalist math teacher.

Environmental Math – By cutting down a forest a logger makes $30.

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[1] Tom Lehrer, a noted math professor and stand up entertainer, came out pretty strongly against the New Math. If you would like to hear his sixties comedy routine that bashes this subject, click here or on the New Math button at pumped101.com

a) Is destroying our forests a good way to make money?
b) How do you think the birds and squirrels feel?
c) Draw a picture of the forest as you would like it to look.

OK, enough about politics. The purpose of this tutorial is to help you understand why we often need rather complex math (calculus) to solve problems in the pump industry but, I promise you that we will not delve into it in this tutorail. More importantly, it will emphasize that the use of less complex math (ie simple algebra) can more clearly explain certain relationships than can words. And finally, the use of simple math can help us learn rather than memorize. And, it is learning, not memorization, that allows us to apply the outcome of one event to yet another.

Newton

Since his name appears in the title of this tutorial, I should probably say something about this fellow. There are many people, both past and present, whose intellect I hold in awe. But, if I were limited to a single choice, it would have to be Sir Isaac Newton.

Newton was born in Woolsthorp, England in 1642 and graduated from Trinity College (Cambridge) in 1664 at the age of twenty-two. This was the year that the Bubonic plague swept through London and, shortly after his graduation, Trinity closed for almost two years. With further studies on hold, Newton decided to return to his family farm in Woolsthorp to ponder what he had learned.

His accomplishments during these two short years, away from the college, represent a phenomenal achievement in human intellect. He formulated the universal law of gravity and the three fundamental laws of motion. He also developed a theory on the nature of light and showed that normal, white light is composed of a rainbow of colors. And yes, he also invented calculus. Unfortunately his works were largely unavailable until 1687 when his book, "Principia Mathematica Philosophiae Naturalis" was published in Latin. And, it was not available in English until 1729 - - two years after his death.

Two numbers make these achievements even more amazing. The first was his age, just 23 at the time. The second was the year - - 1664, several years before the advent of the computer, the internet, the calculator, the slide rule, and much of the math needed to make the slide rule work! It is somewhat sobering for me to admit that I probably could not repeat most of his work even with the knowledge and tools that I have available to me today. But, then again, there is a rather large distinction between knowledge and intellect.

Why Did Newton Invent Calculus?

Necessity is said to be the mother of invention. Now some inventions are accidental - - they can be an unintended byproduct of some totally different endeavor. But, the math invented by Newton was not an accident. It was absolutely necessary if he was to continue his studies. Had it been available during Galileo's time, he might have beaten Newton to the punch.

It has been reported that Newton stumbled upon the law of gravity when an apple fell on his head while sitting beneath the tree. Although this is highly suspect, it is reasonably well documented that seeing an apple fall to the ground led him to consider that the very same force, that caused the apple to fall, held the moon in its orbit and probably extended onwards to infinity. Although his universal law of gravity can be stated in simple algebraic terms, it's derivation was anything but simple.[2] The earth's gravitational attraction (and therefore the weight of an object) is greatest at its surface. Double the distance from its center and gravity is reduced to $\frac{1}{4}$ of that at the surface.   Three times the distance yields just 1/9. Unfortunately the points between these values does not produce a straight line. If they did we could plot any two of those points on a graph, draw a straight line through them, and come up with any value of x and y. But, when the line is a continuously changing curve a different kind of math is required.

Calculus, however, *is* the mathematics of change. It allows us to take an infinite number of points that are nearly zero in value and make some sense of the overall change that is going on. Basically there are two types - - differential (from differentiate) and integral (from integrate). Differential calculus determines the rate of change and will let us to find the velocity or acceleration of some object based on its ever changing position. Integral calculus will determine the quantity where the rate of change is known and allows us, for example, to compute the area or volume of some shape that contains one or more curved surfaces. The derivative and integral are also reciprocals of one another. If you differentiate a mathematical function you can integrate the results to get back where you started!

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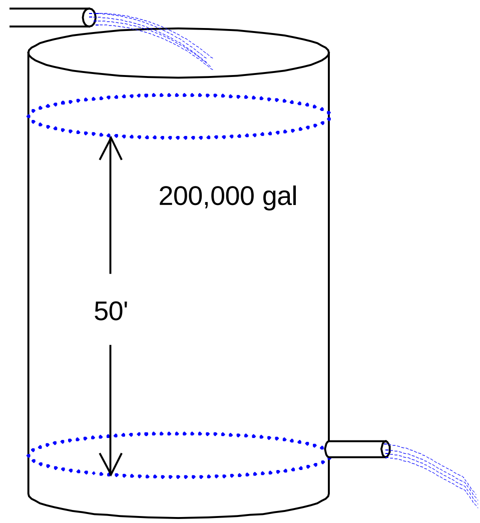
[2] $F = G\, m_1 m_2/d^2$ states that the force one body exerts upon another is directly proportional to the product of their masses and inversely proportional to the distance between them. G is the universal gravitational constant and was not measured until late in the 18th century.

Lets take a look at some practical examples that will illustrate why Newton and, we in the pump industry, occasionally need calculus.

The tank to the right has a capacity of 200,000 gallons between it's upper and lower levels (shown in blue) and the vertical distance between them is 50 feet. The drain at the bottom level exits directly into the atmosphere. A pump supplies the tank with 2200 GPM through the pipe at the top. When the tank is filled to the upper level and the drain valve is fully open, the pump's capacity matches that of the drain so perfectly that the water level in the tank does not change by even a fraction of an inch!



Got all that? OK, here is the question. If the pump is shut off, how long will it take for the tank to empty? Well, this is pretty straight forward. If the pump is producing 2200 GPM then the drain must have the same capacity if the level is to remain the same while both are operating. Since the flow rate is 2200 GPM it should take about 91 minutes to fully drain its 200,000 gallon capacity. No calculus needed here - - just basic arithmetic.

But wait a minute. Doesn't the flow rate of the drain have something to do with the velocity of the water flowing through it? And, isn't there some relationship between velocity and the height of the water above the drain? And, as the height of the water above the drain decreases, will not the velocity and therefore the flow rate also decrease? The answer is yes and, if I remember correctly, that relationship takes the form of $v^2 = 2gh$, where v is the velocity of the water, g is the force of gravity at 32 ft/sec^2, and h is the height of the water column.[3] Darn, good old arithmetic just got replaced with algebra!

This simple algebraic formula can predict the velocity of water in the drain for any given elevation of water in the tank. For example when the tank is full the velocity of the exiting water is:

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[3] See the "Up and Down" Puzzler for a more detailed review of initial and final velocity.

$v^2 = 2 * 32\ ft/sec^2 * 50\ ft$ $v^2 = 3200\ ft^2/sec^2$ $v = 56.56\ ft/sec$

So, at a flow rate of 2200 GPM, the velocity is a little over 56.5 ft/sec. I never mentioned the diameter of the drain but based on these two numbers we can use simple algebra to calculate its size. 2200 GPM is equivalent to 36.68 GPS (gallons per second) and 1 GPS is equal to 0.1337 ft^3 of water per second. Therefore the initial flow through the drain is 4.9 ft^3/sec. The simple equation below will give us the drain size in square feet.

ft^2 = volume per sec / velocity $ft^2 = 4.9\ ft^3/sec\ /\ 56.5\ ft/sec$ $ft^2 = 0.087$

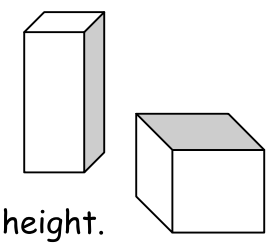
And, 0.087 square feet is equal to 12.5 square inches which almost exactly the cross sectional area of a 4" pipe. See, I told you that simple math can be useful. Now, we would not normally use such a small drain on a tank of this size but I wanted the flow to be somewhat restricted so that I can illustrate what happens at various points in the draining process.

When the tank is half full (25 ft) the velocity drops to 40 ft/sec ($v^2=2gh$) and since it's change is proportional to the change in flow, flow drops a similar percentage to about 1557 GPM. It appears then, that what we really need is the average velocity from top to bottom of the tank. If we had that figure we could convert it to average flow and come up with the time required for the entire 200,000 gallons to exit the tank.

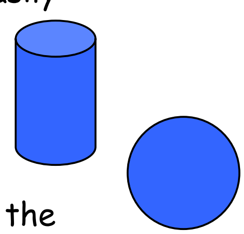
Now, we could get an "approximate" average velocity by computing the values of v at several levels between full and almost empty. Our approximation would be more accurate if we measured v at every foot and even more accurate if we measured v at every inch. But, I suspect that you see where we are going here. We could never compute the true, average velocity with algebra, because regardless of how small our measurement of h, the average will always be approximate. It requires calculus, and its ability to derive an infinite number of values for h and v, if we are to obtain a true average. Based on my "somewhat rusty" calculus, I came up with a drain time of 182 minutes which equates to an average flow of 1099 gpm and an average velocity of 28.22 ft/sec. But forget averages for a minute. These calculations also show that the level went from 50 to 49 feet in less than two minutes but, the last foot took almost 25 minutes! Try "straight lining" that one.

Did I hear a "so what" out there? After all, if we really need to know how long it takes a tank to drain then just drain the dang thing! All it takes is a stopwatch and someone to start and stop it. But suppose, for a moment, that we are designing that tank to supply water to a town or some other liquid to a process line. If we know the minimum flow that must be maintained, we can calculate how long a particular design can provide it. Even though the volume of the tank in our example may be adequate, its flow may not. Depending upon the application we may have to make changes in it dimensions. For example, if we reduce its diameter while increasing its height we could maintain the volume while extending the time that it provides some minimum flow.

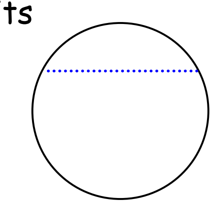
OK, lets take a look a several more examples but this time our question concerns the volume of the tank and not the time it takes to empty. Calculating the volume of a container is pretty straight forward as long as it is made up of straight lines that are parallel with one another. The two tanks to the right fit these requirements. One has sides that are rectangular in shape while the other is a cube that consists of squares. The volume of each is determined by a simple equation that takes the area of the base of the tank and multiplies it by its height. Simply stated - - Volume = Length x Width x Height. Turn them on their sides and the same equation still applies. Another important consideration is that the volume of liquid in the tank is always directly proportional its level. $\frac{3}{4}$ of the way up it is $\frac{3}{4}$ full and $\frac{3}{4}$ of the way down it is $\frac{1}{4}$ full. Again, this holds true even if they are turned on their sides. Above ground tanks of this design are usually limited to relatively small volumes because of the pressure the liquid exerts on their flat, vertical surfaces. Larger ones are typically designed for underground storage where soil provides the foundation to support the walls.



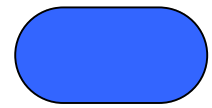
Thanks to the pre-calculus work of Euclid (born 325 BC), we can also easily calculate the volume of cylindrical and spherical tanks. The reason we can work with these curved surfaces without calculus is that they are made up of circles. And, the beauty of a circle is that its curvature is constant. This led to the discovery that the ratio of a circle's diameter to its circumference never changes regardless of the size of the circle. This ratio is known as Pi (π) and has an approximate value of 3.1416.



To calculate the volume of a cylinder all we need to know is the area of its circular shape, which is πr^2 (where r is the radius or $\frac{1}{2}$ the diameter) and its height which is h. In simple math - - volume = $\pi r^2 h$. When in the upright position, the volume of liquid in the tank is always directly proportional to its level. But, unlike rectangular tanks, if you turn it on its side this proportionality goes away (with the exception of full, $\frac{1}{2}$ full, and empty) and calculating its volume becomes extremely complex. The example on the

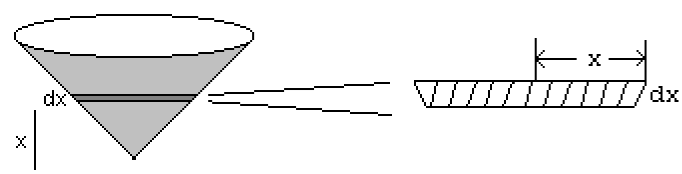


right illustrates the complexity of the resulting surface. This lack of proportionality also holds true for a sphere (volume = $4/3\ \pi r^3$) because its dimensions are the same in all directions. I guess you could say that it is always on its side even though it doesn't have sides. Want one that is even more complex? Consider the common propane tank (shown to the right) that is used to store heating fuel for homes. It is not just a horizontal cylinder, but also has ends that are hemispherical! You will find a number of calculators on the web that will allow you to calculate the partial volumes of horizontal cylinders and spheres.



My dad was a gas and fuel oil distributor when I was growing up and he had many overhead, horizontal tanks at his storage facility. This was well before the advent of web based calculators (or even the hand held calculator for that matter) so they had to rely on another method of determining the volume of a partially filled tank. He employed a long measuring stick to obtain the distance of the fuel level from the bottom of the tank and then used a table, provided by the tank manufacturer, to get the remaining volume in gallons.

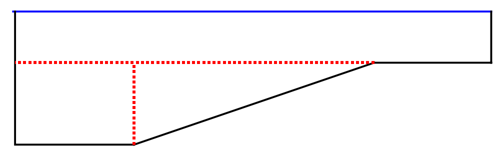
Well, so far we have managed to avoid calculus when it comes to computing the volume of a tank. Obviously there must be some tanks that require calculus or I would not have written this tutorial.
Consider the conical tank to the right. It is much like a cylindrical tank except that its diameter is increasingly larger as it goes from bottom to top. Think of it as an infinite number of circles of infinitely small increasing diameters stacked on top of one another. Actually that is pretty hard to imagine because infinity, whether large or small, is beyond our normal abilities of comprehension. But, it should cause you to think back to the basic problem we had in trying to compute the time it takes for a cylindrical tank to drain. There, velocity changed with respect to infinitely small changes in height. So did volume but it was always directly proportional to height. In the case of a cone, volume is not directly proportional to height because as height increases so does its diameter. With the help of calculus, we obtain $v = 1/3\pi r^2 h$ - - a simple solution to a complex problem.



Now I will have to admit that, with the exception of the martini glass, cone shaped tanks are seldom used to store liquids. More often they are used to measure solids (powders & aggregates) into some process stream but it is a good example none the less. Lets take a look at a more practical example.

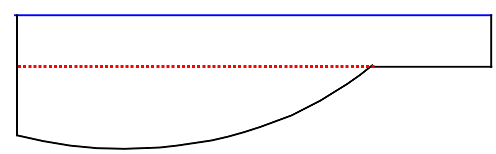
The swimming pool is nothing more than a tank which may reside inside or outside and in both above and in ground configurations. The flow rate of its recirculation pump depends upon the turnover time. Turnover time is defined as the time required to recirculate the total capacity of the pool through the filtering system. Turnover time is subject to local regulations but can vary from twelve hours for residential pools to as little as four hours for public facilities. In order to obtain the flow rate necessary to meet some turnover time we need to know the capacity of the pool and hence my purpose for bringing this up in the first place.

Take a look at the outline of the pool on the right. It is a length wise cross section with the water level represented in blue and its dimensions shown in black. As long as we know, or can measure, its width and the black lines we can compute its volume. The red dotted lines show one way (there are several) that we can determine the cross sectional area of the pool.

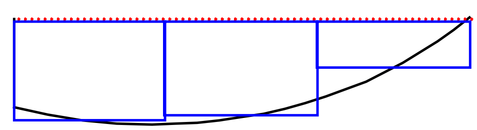


The areas of the two rectangles are simply height x length and the right triangle is ½ its base x height. Add the three areas together and multiply the result by the width of the pool and you get the volume in cubic feet which is easily converted to gallons.

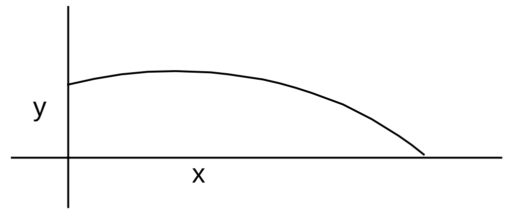
The pool to the right has the same overall dimensions as the one above but there is one very big difference. The portion that goes from the shallow end of the pool to the deep end is no longer a straight line. It is now a continuous curve. We can still compute the area of the rectangle but how do we come up with the area of the curved portion below?



One method uses a series of rectangles to approximate the area. The figure to the right shows the curved section enlarged and covered by three blue rectangles. One adjusts the height of the rectangles so that the portion that falls below the curve fills the missed portion above the curve. Add the individual areas of the rectangles and you get an approximation of the curved area. The narrower, and more frequent the rectangles, the closer the approximation becomes.



We used the rectangle method of approximation quite a bit back during my undergraduate days in our chemistry and physics labs. Most of the scientific instruments (chromatographs, etc) plotted smooth curves with respect to time and often, all we needed to know was the approximate concentration of the compound represented by the curve. The quality and consistency of the chart paper in the sixties provided an even simpler method of approximation. We would carefully cut out the curves and weight them on an analytical balance!



But, if you want to compute the true area of the curved portion it takes calculus. Just flip the curve over and calculus will measure the value of y at infinitely small values of x. If fact, the method used by calculus is quite similar to that of the rectangle approximation. The difference is that the rectangles used by calculus are infinitely narrow. Taken together they will represent the true area under the curve. Once that area is determined it is multiplied by the width of the pool to obtain its volume. Add this volume to the volume of the upper, rectangular portion and you have the total capacity of the pool.

Want another quick example. The volume or capacity of a piston pump can easily be calculated using the same simple math we used to calculate the volume of a cylindrical tank. But have you ever seen the rotor and stator of a progressing cavity pump? It takes calculus even to approximate its capacity!

Hopefully this "amathematical" introduction to calculus has given you a better understanding of its capabilities and some of its applications in our industry. It is really not as difficult as it is made out to be and if you are interested in investigating it a little further, there is a great beginners course at

http://vps.arachnoid.com/calculus/index.html .

Even if you decide not to delve into calculus, I hope that you will download "The Puzzler" and take a look at how simple algebra can help explain some of the often, complex physics of hydraulics and mechanics. One Puzzler, for example, compares initial and final velocity and explains why the velocity of water leaving a pump's impeller is the same as if it fell from the height it actually attains. Another shows that the Bernoulli equation is really all about conservation of energy and that it not only describes the relationship of velocity and pressure in a flowing pipe but also why airplanes fly and baseballs curve. Others discuss buoyancy, hydrostatic pressure, centrifugal force, friction, corrosion, and electricity. In all there are more than forty of these "brain teasers" that can help you further your knowledge of pumps, motors, and controls.